A clique graph based merging strategy for decomposable SDPs

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Semidefinite programming

- Given matrices $C, A_1, \ldots, A_m \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$, find X:

$$\begin{array}{ll} \text{minimize} & \langle C, X \rangle \\ \text{subject to} & \langle A_i, X \rangle = b_i,\ i = 1, \ldots, m \\ & X \in \mathbb{S}^n_+ \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^\top \nu \\ \text{subject to} & A^\top \nu + Y = C \\ & Y \in \mathbb{S}^n_+. \end{array}$$

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$\mathcal{O}(n^3)$

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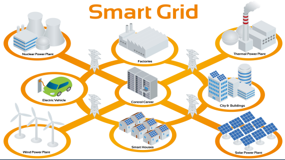
$n = 10^4 \rightarrow \mathcal{O}(10^{12})$ operations at each step

Semidefinite programming

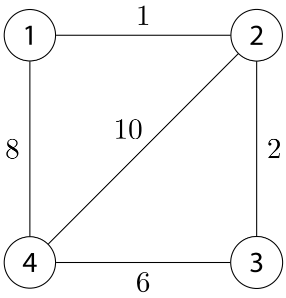
Where do we find positive semidefinite matrices?

- Lyapunov functions
- Linear Matrix Inequalities / S-procedure [Martin S Andersen et al. 2014]
- Kernel matrices [Lanckriet et al. 2004]
- Covariance matrices [Bertsimas and Nino-Mora 1999]
- Graph Laplacian
- Sum-of-Squares [Lasserre 2009]
- Semidefinite relaxation of
  - cardinality constraints (sparse PCA) [d'Aspremont et al. 2004]
  - QCQPs
  - mixed-integer constraints [Goemans and Williamson 1995]



The MAXCUT problem

- Weighted graph $G(V, E)$ with weights $w_{ij} \geq 0$, find $S \subset V$ such that the edge weights between S and $\bar{S} = V \setminus S$ are maximized



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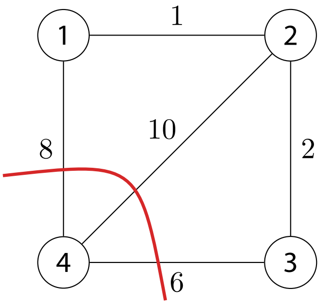

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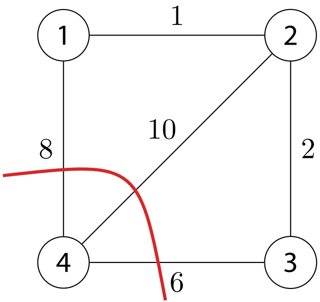

$$\begin{array}{ll} \text{maximize} & \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(1 - y_i y_j) \\ \text{subject to} & y_i \in \{-1, 1\}, \ \forall i \in V \end{array}$$

- NP-hard, part of Karp's 21 NP-complete problems [Karp 1972]
- best approximation until 1995: $0.5p^*$

The MAXCUT problem: A semidefinite relaxation

- SDP relaxation that guarantees $0.87856\, p^*$ [Goemans and Williamson 1995]

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→ Lifting: $X = yy^\top$

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Primal SDP

maximize $\frac{1}{4}\langle L, X \rangle$

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$$\begin{array}{ll} \text{minimize} & \sum_i \nu_i \\ \text{subject to} & Y = \text{diag}(\nu) - \frac{1}{4}L \\ & Y \succeq 0 \end{array}$$

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Overview

Matrix Sparsity and Graphs

Chordal decomposition

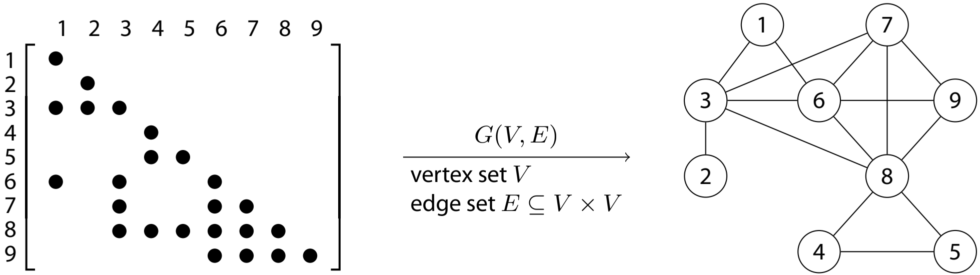
Clique merging

- Clique tree-based merging strategies
- Clique graph-based merging strategy

Benchmarks

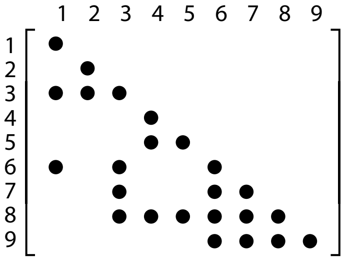
Conclusion

Matrix sparsity and graphs

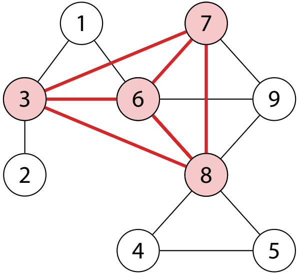


*Survey paper [L. Vandenberghe and M. S. Andersen 2015]

Matrix sparsity and graphs



Complete subgraph



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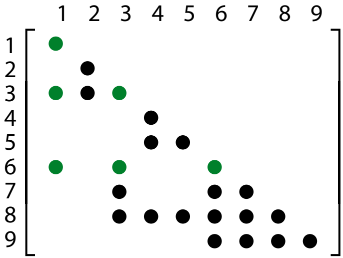
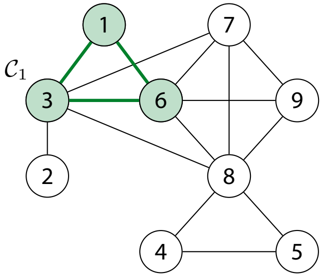


Clique 1



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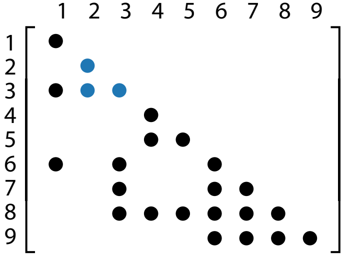
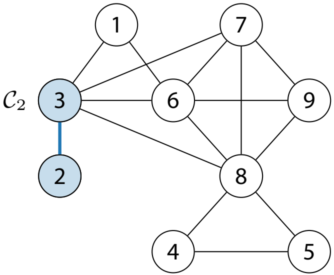


Clique 2



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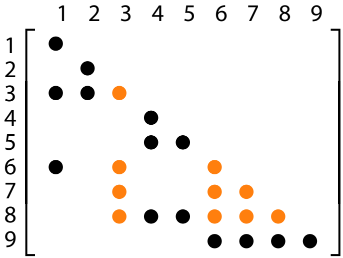
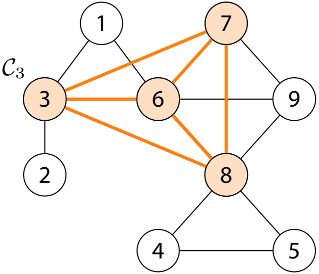


Clique 3



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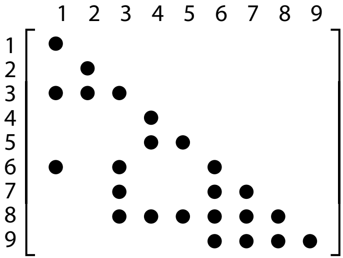
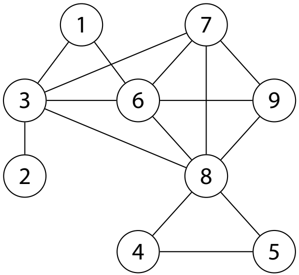


Chordal graph



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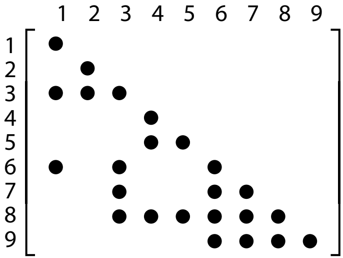

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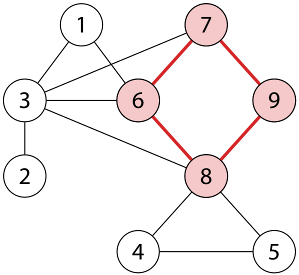

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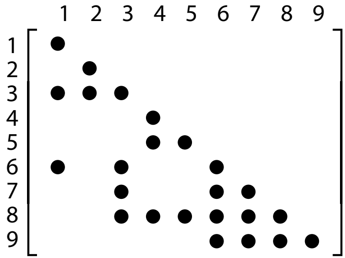

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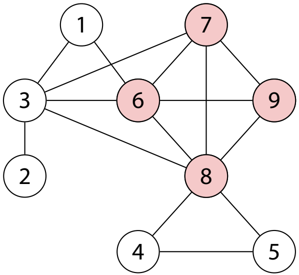

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Chordal decomposition

Dual SDP*

maximize $b^\top y$

subject to $\sum_{i=1}^{m} A_i y_i + S = C$

$S \in \mathbb{S}_+^n$

Sparse matrix cones

$$\mathbb{S}^n(E,0) := \{S \in \mathbb{S}^n \mid S_{ij} = S_{ji} = 0, \text{ if } i \neq j, (i,j) \notin E\}$$
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Agler's theorem

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Let $G(V,E)$ be a chordal graph with a set of maximal cliques $\{\mathcal{C}_1,\dots,\mathcal{C}_p\}$. Then $S \in \mathbb{S}^n_+(E,0)$ if and only if there exist matrices $S_\ell \in \mathbb{S}^{|\mathcal{C}_\ell|}_+$ for $\ell = 1,\dots,p$ such that

$$S = \sum_{\ell=1}^{p} T_\ell^\top S_\ell T_\ell.$$

*[Agler et al. 1988], Grone's theorem for primal form SDPs [Grone et al. 1984]

Example: Chordal Decomposition

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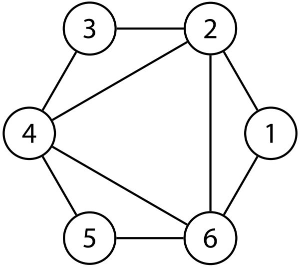

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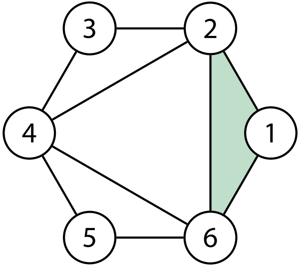

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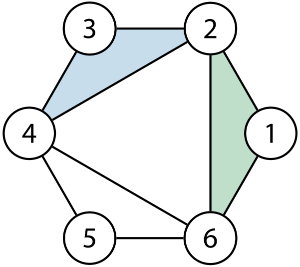

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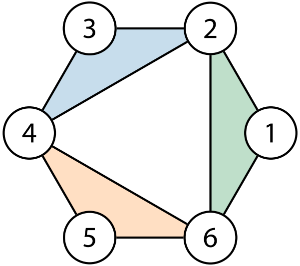

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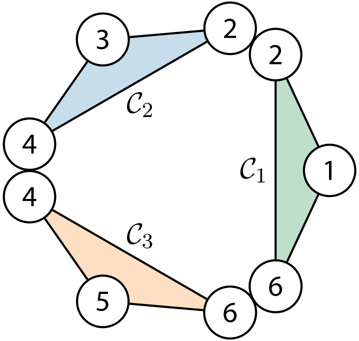

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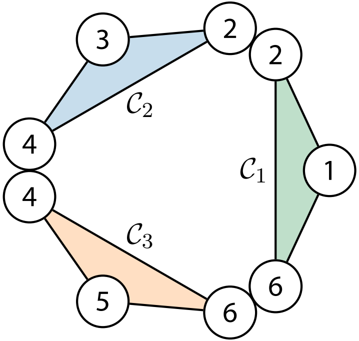
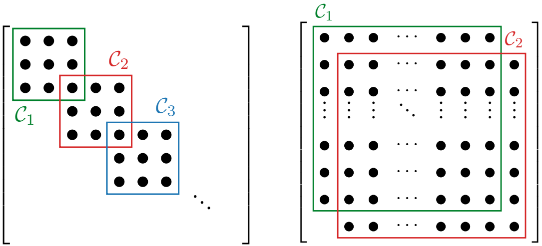


- Interior point method [Fukuda et al. 2001]
- First-order method [Sun, M. S. Andersen, and L. Vandenberghe 2014]
- ADMM-HSDE [Zheng et al. 2019]

Clique merging

- Combine cliques by introducing new edges in the graph
- One merge operation:
  - replaces two PSD constraints by one larger PSD constraint
  - removes equality constraints

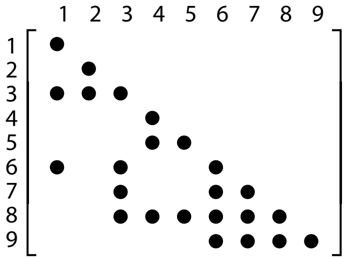
  $\rightarrow$ trade-off depends on the employed solver algorithm
- Obvious cases:

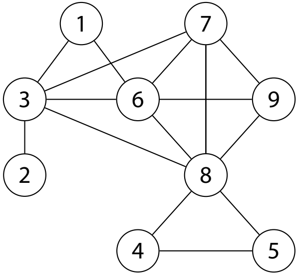


Algorithm: First-order solver

Factor constraint matrix;
while not converged:
[...]
Eigenvalue decomposition of PSD decision variables;

Clique tree-based merging strategies





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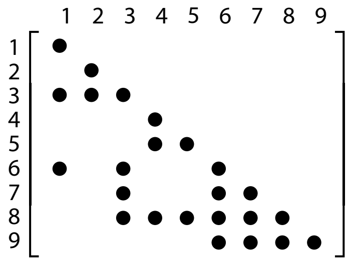

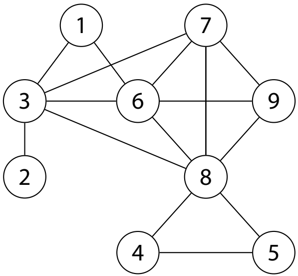

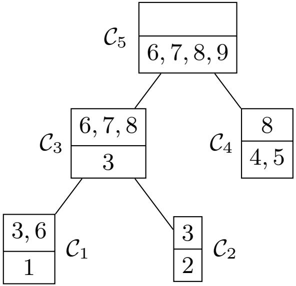

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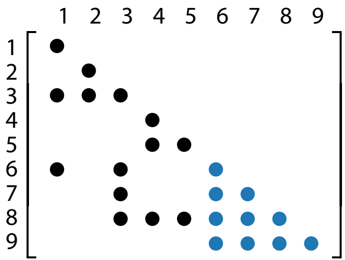

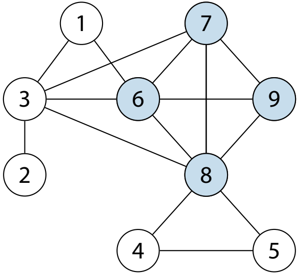

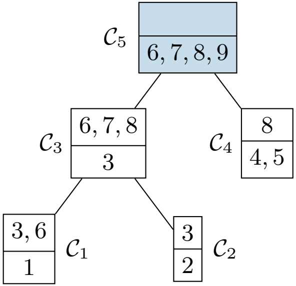

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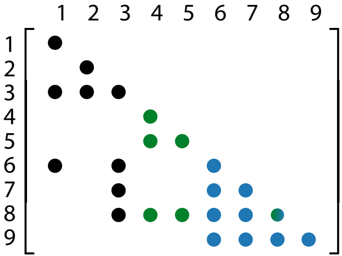

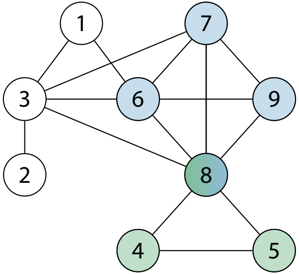

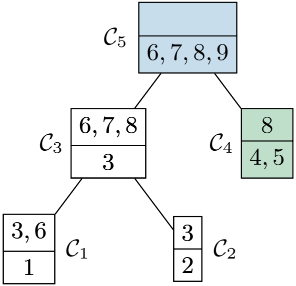

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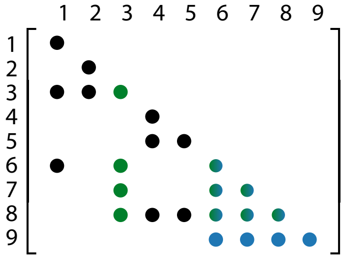

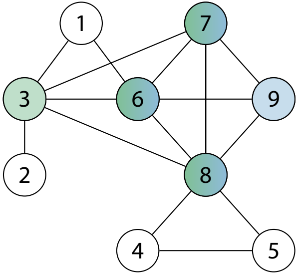

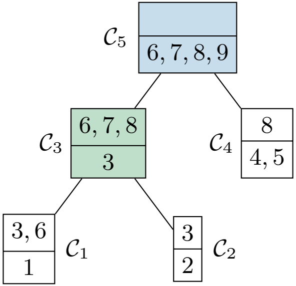

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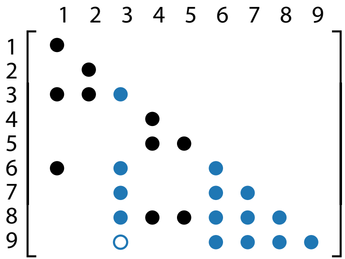

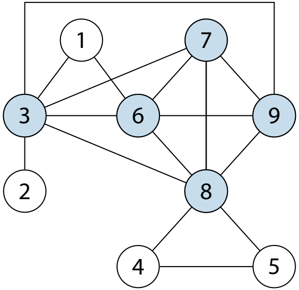

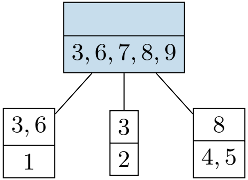

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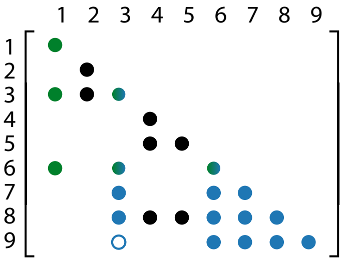

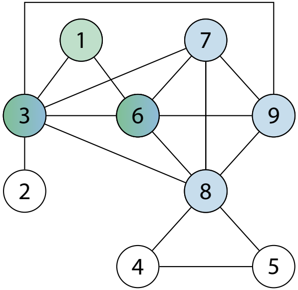

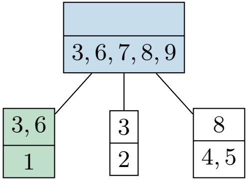

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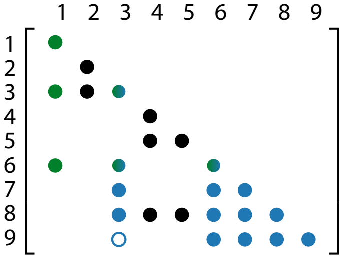

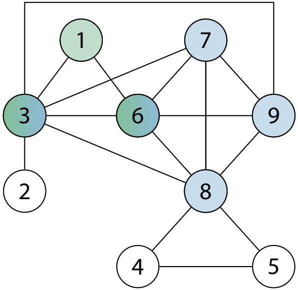

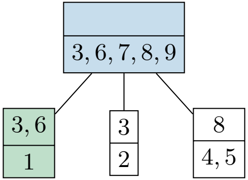

- Designed for interior-point solvers

+ Clique tree cheap to compute and evaluate

– Disregards distant merge candidates

– Relies on heuristic parameters

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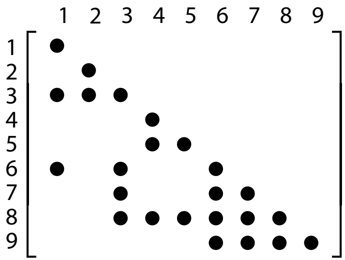
A new clique graph-based merging strategy



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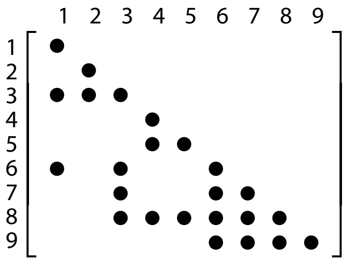

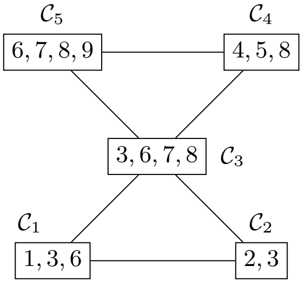

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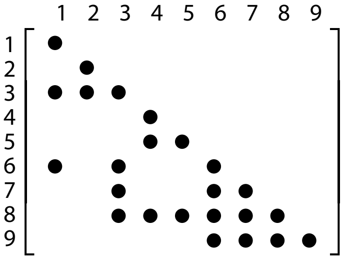

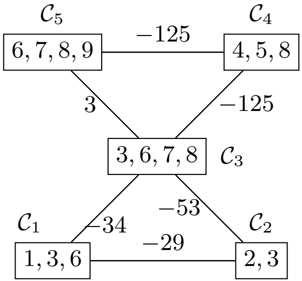

in this example:
$e(\mathcal{C}_i, \mathcal{C}_j) = |\mathcal{C}_i|^3 + |\mathcal{C}_j|^3 - |\mathcal{C}_i \cup \mathcal{C}_j|^3$

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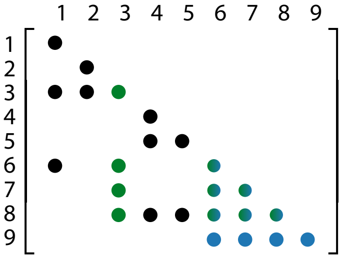

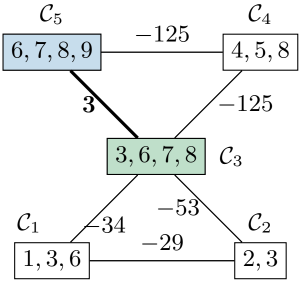

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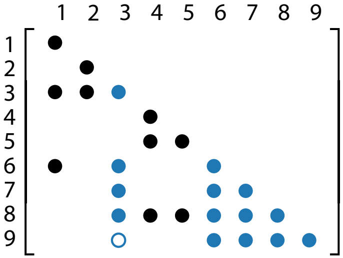

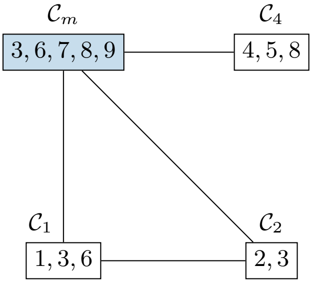

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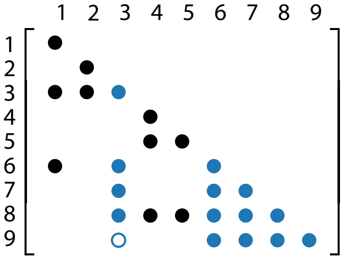

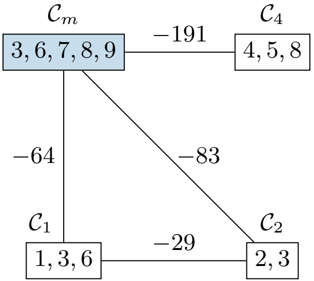

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  Update edge weights connected to $\mathcal{C}_m$;

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- **Problem set:** Large sparse SDPs from the SDPLib collection and SuiteSparse Matrix Library
- **Setup:** Compare different merge strategies with our solver

a) No decomposition	b) No merging
c) `SparseCoLO` merging	d) Parent-child merging
e) Clique graph merging (nominal)	f) Clique graph merging (estimated)

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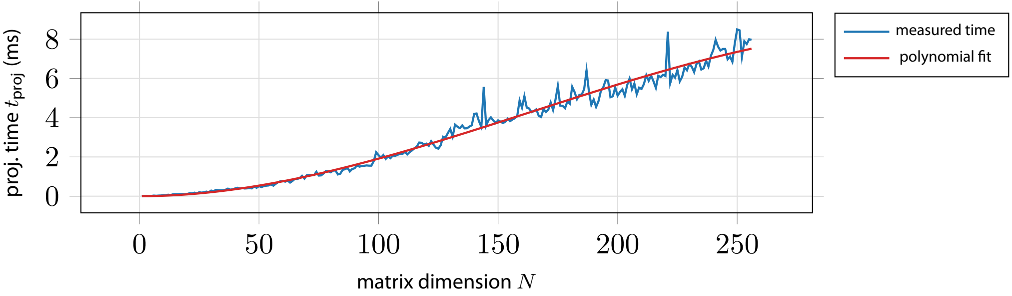
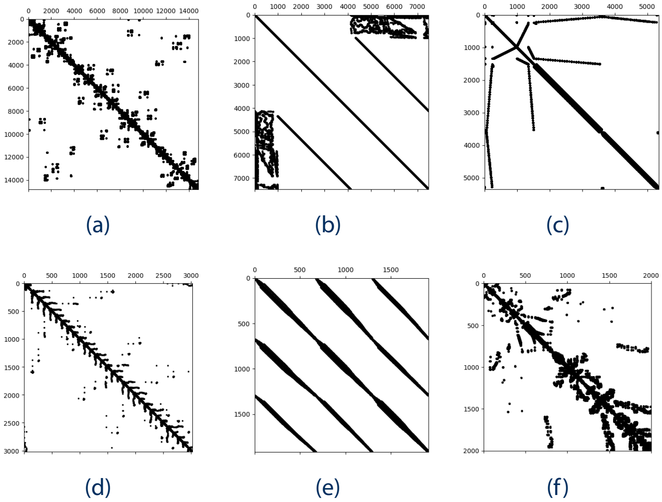
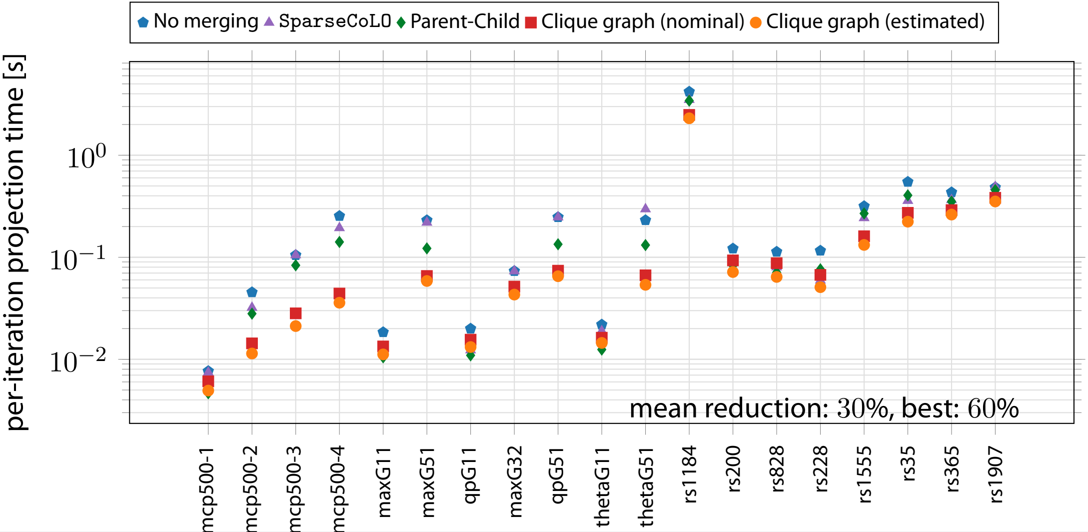


Benchmark sparsity patterns

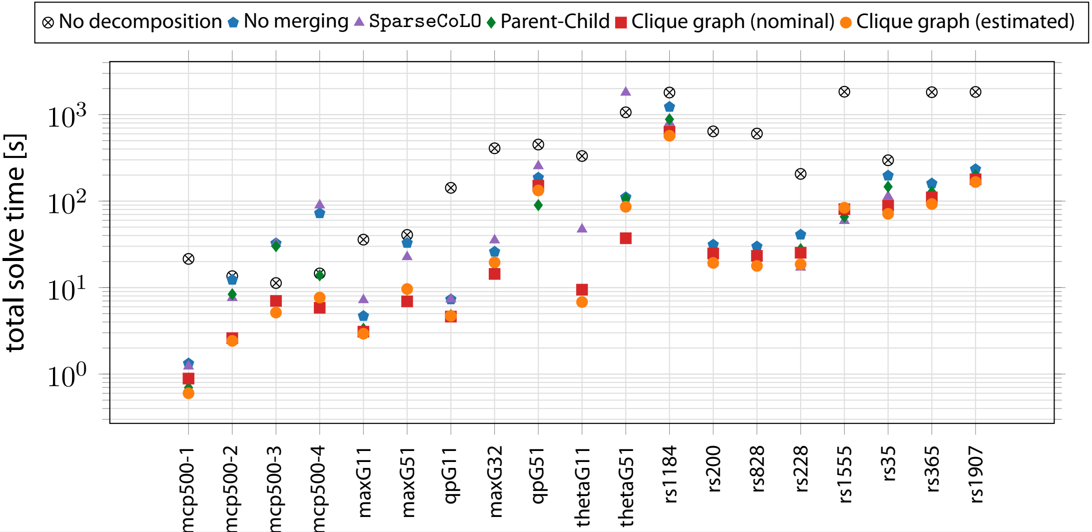
- 500 - 21M nonzeros



Benchmark results - projection time



Benchmark results - solve time



Benchmark results

- Hardware: Oxford ARC-HTC 16 logical Intel Xeon E5-2560 cores, 64GB RAM

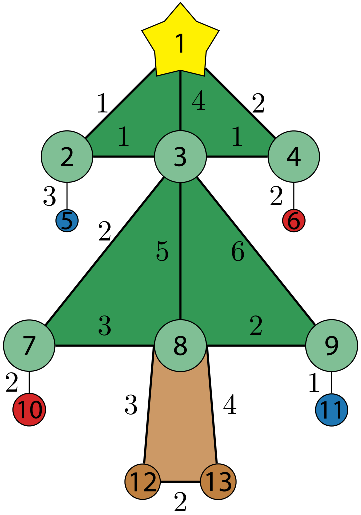
Table: Solve times for different SDP solvers.

problem	COSMO	Mosek	SCS	problem	COSMO	Mosek	SCS
maxG11	**1.47**	4.45	131.8	rs1184	**224.86**	$*\,*\,*^m$	$*\,*\,*^m$
maxG32	**6.25**	50.84	840.79	rs1555	**66.6**	$*\,*\,*^\dagger$	$*\,*\,*^m$
maxG51	**8.09**	9.92	36.56	rs1907	**104.61**	$*\,*\,*^\dagger$	$*\,*\,*^m$
mcp500-1	**0.24**	1.7	29.28	rs200	**12.47**	752.27	$*\,*\,*^\dagger$
mcp500-2	**1.68**	1.75	17.36	rs228	**12.86**	395.24	982.5
mcp500-3	4.41	**1.68**	8.36	rs35	**54.88**	919.19	$*\,*\,*^\dagger$
mcp500-4	8.2	**1.76**	7.4	rs365	**62.65**	$*\,*\,*^\dagger$	$*\,*\,*^\dagger$
qpG11	**2.36**	26.23	734.7	rs828	**10.84**	825.03	$*\,*\,*^\dagger$
qpG51	121.6	**96.42**	527.55	thetaG51	71.21	**50.08**	967.43
thetaG11	**2.32**	8.53	142.53				

^{m} out-of-memory error, † 30min timelimit

Code example

What is the maximum cut through this Christmas tree?



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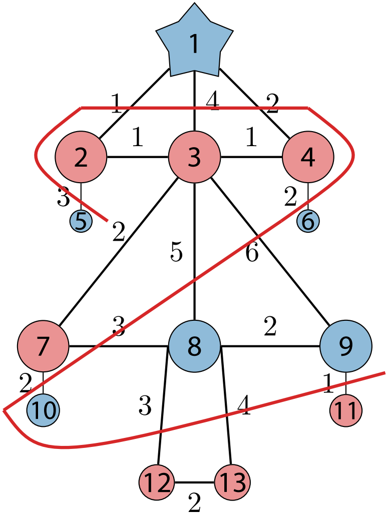

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